Exam Advanced Quantum Mechanics

Before you start, read the following:

- There are 4 problems for a total of 40 points.
- Start each problem on a new sheet of paper.
- Write your name and student number on each sheet of paper.
- Illegible handwriting will not be graded.
- *Good luck!*

Problem 1 (*45 minutes; 10 points in total*)

Consider a tritium atom, made up of a triton ^{3}H nucleus with $Z = 1$ and an electron in its ground state with $n = 1$, $\ell = 0$, $m = 0$. Suppose the triton undergoes β decay,

$$^3\text{H} \rightarrow {}^3\text{He} + e^- + \overline{\nu} \ ,$$

so that the triton turns into a helion nucleus and the nuclear charge suddenly increases to $Z = 2$. The β electron is emitted with high energy $\sim$ 15 keV and leaves the atom very rapidly. Thus, at the time t_0 of the decay an ionized $^3\text{He}^+$ atom is formed and the wave function of the remaining bound electron at t_0 is still the same as in tritium.

3 pnts (a) Give the Hamiltonian H_1 of the atomic electron before the decay and the Hamiltonian H_2 of this electron after the decay (when the potential energy has suddenly changed). What are the Bohr energy levels of the electron in the $^3\text{He}^+$ ion in Rydberg, its Bohr radius, and its ground-state wave function?

2 pnts (b) Give an expression for the probability $P(n, \ell, m)$ that the atomic electron ends up in the state $|n, \ell, m\rangle$ of $^3\text{He}^+$ after the decay. Show that only the probabilities $P_n = P(n, 0, 0)$ are nonzero.

3 pnts (c) Calculate the probability P_1 that the atomic electron ends up in the ground state of $^3\text{He}^+$. Show that it is $P_1 = 0.70$.

2 pnts (d) Calculate the total probability that the electron ends up in any bound state of $^3\text{He}^+$. Use

$$\sum_{n=2}^{\infty} P_n = 0.27 \ .$$

Explain why it does not equal 1.

The ground-state wave function for a hydrogen-like system is

$$\psi_{n=1,\ell=0,m=0}(\vec{r}) = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0} \ , \quad a_0 = \frac{\hbar}{\alpha m_e c} \ .$$

Problem 2 (*45 minutes; 10 points in total*)

2 pnts (a) The q components ($q = -k, \ldots, k$) of an irreducible spherical tensor of rank k can be written in terms of spherical harmonics $Y_\ell^m(\theta, \phi) = Y_\ell^m(\hat{n})$ as

$$T_q^{(k)} = Y_{\ell=k}^{m=q}(\vec{V}) \ .$$

Give the components $V_m^{(1)}$ of a spherical tensor of rank 1 (that is, a vector), in terms of the cartesian components. Give the spherical components of $\vec{r} = (x, y, z)$.

3 pnts (b) Give the Wigner-Eckart theorem for the matrix element of a tensor operator with respect to angular-momentum eigenstates in formula form. Formulate in words what it implies and in which way it is useful.

3 pnts (c) Consider a spinless particle bound to a fixed center by a central force potential. Relate, as much as possible, the matrix elements

$$\langle n', \ell', m'| \mp \frac{1}{\sqrt{2}}(x \pm iy)|n, \ell, m\rangle \quad \text{and} \quad \langle n', \ell', m'|z|n, \ell, m\rangle \ ,$$

by using *only* the Wigner-Eckart theorem. State explicitly when the matrix elements are nonzero.

2 pnts (d) Do the same problem as in (c) for the wave functions

$$\psi(\vec{r}) = R_{n\ell}(r) Y_\ell^m(\theta, \phi) \ .$$

$$\begin{aligned} Y_1^0 &= \sqrt{\frac{3}{4\pi}} \cos\theta \\ Y_1^{\pm 1} &= \mp\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi} \end{aligned}$$

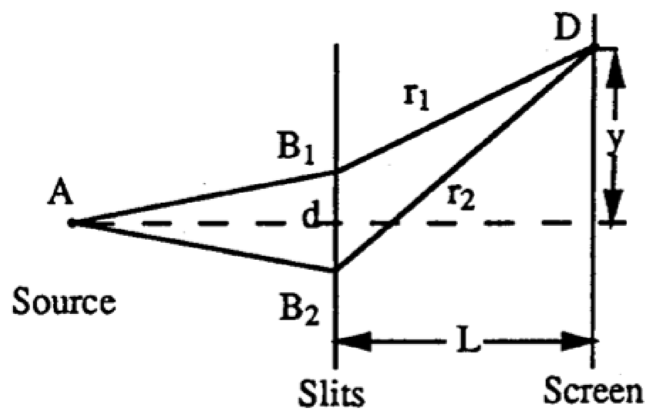
Problem 3 (*45 minutes; 10 points in total*)

In the path-integral formalism, the probability for a particle to go from position $\vec{r}_1$ at time t_1 to position $\vec{r}_2$ at time t_2 is given by $P(2,1) = |U(2,1)|^2$, where the propagator $U(2,1)$ is given by the "sum over paths"

$$U(2,1) = N_{21} \sum_a \exp\left[iS_a(t_2,t_1)/\hbar\right] \ ,$$

where N_{21} is a normalization constant and $S_a(t_2,t_1)$ is the classical action for path a from $x_1 = (\vec{r}_1, t_1)$ to $x_2 = (\vec{r}_2, t_2)$.

2 pnts (a) Show if a path from x_1 to x_2 consists of two successive segments with an intermediate point x_3 that $U(2,1) = \int U(2,3)U(3,1)\mathrm{d}^3\vec{r}_3$.



3 pnts (b) Consider the two-slits experiment in the Figure. Assume that only the classical paths need to be taken into account. Give an expression for $U(D,A)$. Use the notation $U(2,1) = N_{21}\exp(i\phi)$ for the free-particle propagator.

3 pnts (c) Show that the probability distribution (intensity pattern) on the screen reads $P(D,A) = 2I(1+\cos\theta)$ and give expressions for I and for θ. Discuss the interference pattern on the screen.

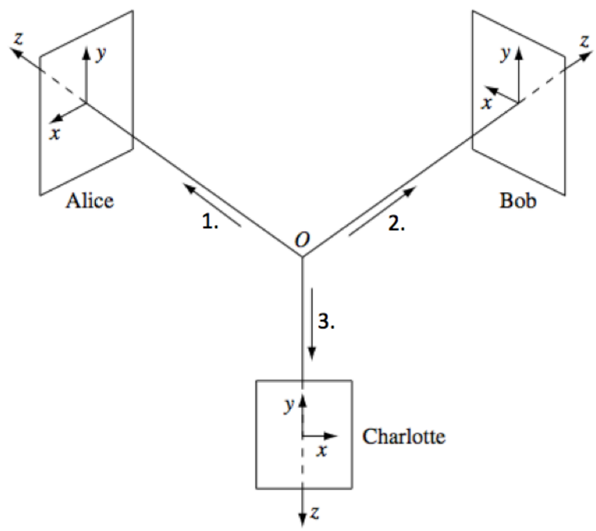
2 pnts (d) Show that $\theta = p|r_1 - r_2|/\hbar$, where $p = m(r_1+r_2)/2\tau$ is the mean momentum. From considering the first maximum, show that the relation $\lambda = h/p$ holds.

The propagator for a free particle of mass m is given by

$$U(2,1) = \left[\frac{m}{2\pi\hbar i(t_2-t_1)}\right]^{3/2} \exp\left[\frac{im}{2\hbar}\frac{(x_2-x_1)^2}{t_2-t_1}\right] \equiv N_{21}\exp(i\phi) \ .$$

Problem 4 (*45 minutes; 10 points in total*)

Consider a central source that sends particles to three measuring devices, operated by Alice, Bob, and Charlotte, which are space-like separated from each other. Each device has two settings, labelled X and Y, and records events by an "up" or "down" result. The records will thus have the label X_1, X_2, X_3, Y_1, Y_2, or Y_3, and each will give a result $+1$ or -1.



2 pnts (a) The experimentalists discover that certain products are not random. They find that, whatever the individual recordings, $X_1Y_2Y_3 = Y_1X_2Y_3 = Y_1Y_2X_3 = +1$. Argue how in classical mechanics such correlations could occur and show that $X_1X_2X_3 = +1$.

In fact, the observables measured are $X_i = \sigma_x^{(i)}$, $Y_i = \sigma_y^{(i)}$ $(i = 1, 2, 3)$, where $\vec{S}^{(i)} = \hbar\,\vec{\sigma}^{(i)}/2$ is the spin vector of particle i. The source sends out the three particles in the state

$$\psi(1,2,3) = \left(\chi_\uparrow^{(1)}\chi_\uparrow^{(2)}\chi_\uparrow^{(3)} - \chi_\downarrow^{(1)}\chi_\downarrow^{(2)}\chi_\downarrow^{(3)}\right)/\sqrt{2}\ ,$$

where $\chi_\uparrow^{(i)}$ and $\chi_\downarrow^{(i)}$ are the eigenstates of $\sigma_z^{(i)}$ with eigenvalues $+1$ and -1.

3 pnts (b) Calculate $X_i\,\psi(1,2,3)$ and $Y_i\,\psi(1,2,3)$ for $i = 1, 2, 3$. Check the results for $X_1Y_2Y_3$, $Y_1X_2Y_3$, and $Y_1Y_2X_3$ given above.

2 pnts (c) Calculate $X_1X_2X_3\,\psi(1,2,3)$.

3 pnts (d) Discuss the implications of these findings for the Einstein-Podolsky-Rosen problem, similar to the Bell inequalities for two particles.